PACKING SETS AND THEIR VARIATIONS IN GRAPHS

A SYNOPSIS

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1 Introduction

Packings, colorings and their related concepts are actively scrutinized areas of research in graph theory at the moment due to their extensive applicability in practical situations such as coding theory [1], biological networks [2], neural networks [3], scheduling a Packet Radio Network(PRN) [4] and so on.

More specifically, an injective edge coloring partitions the edge set into subsets. In order to analyze these subsets, we initiate the parameter "Edge open packing number" of graphs. An edge e of a graph G is a *common edge* of two distinct edges e_1, e_2 different from e if e joins a vertex of e_1 to a vertex of e_2. That is, $\langle e_1, e, e_2\rangle$ forms either P_4 or C_3. A non-empty set $D \subseteq E(G)$ is an *edge open packing set* if no two edges in D have a common edge in G. The maximum cardinality among all edge open packing sets is the *edge open packing number* of G and is denoted by $\rho_e^o(G)$. An edge open packing set of cardinality $\rho_e^o(G)$ is called a ρ_e^o-set of G.

Next, we analyze the edge open packing number when a vertex or an edge is removed from the graph. At this stage, it may increase or decrease, or remains unchanged. As a result, we get $V(G) = V^0(G) \cup V^-(G)$ and $E(G) = E^-(G) \cup E^0(G) \cup E^+(G)$, where

$$\begin{aligned}
V^0(G) &= \{v \in V(G) : \rho_e^o(G - v) = \rho_e^o(G)\}\\
V^-(G) &= \{v \in V(G) : \rho_e^o(G - v) < \rho_e^o(G)\}\\
E^-(G) &= \{e \in E(G) : \rho_e^o(G - e) < \rho_e^o(G)\}\\
E^0(G) &= \{e \in E(G) : \rho_e^o(G - e) = \rho_e^o(G)\}\\
E^+(G) &= \{e \in E(G) : \rho_e^o(G - e) > \rho_e^o(G)\}
\end{aligned}$$

Further, we introduce the new variations corresponding to open packing namely the "Open packing subdivision number" and the "Outer-connected open packing number" of graphs. A non-empty set $S \subseteq V(G)$ of a graph $G = (V, E)$ is an *open packing set* of G if no two vertices of S have a common neighbor in G. The maximum cardinality of an open packing set is called the *open packing number* of G and is denoted by $\rho^o(G)$.

Minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the value of $\rho^o(G)$ is called the *open packing subdivision number* of G and is denoted by $sd_{\rho^o}(G)$. Now, we categorize the graphs into three classes regarding the open packing subdivision number as follows.

Class A: Graphs with $sd_{\rho^o}(G) = 1$.
Class B: Graphs with $sd_{\rho^o}(G) = 2$.
Class C: Graphs with $sd_{\rho^o}(G) = 3$.

In a connected graph G, an open packing set S with the property that either $S =$

$V(G)$ or $\langle V - S\rangle$ is connected is called an *outer-connected open packing set* (*ocop-set*) of G. The minimum and maximum cardinalities of an outer-connected open packing set are respectively called the *lower outer-connected open packing number* and the *outer-connected open packing number* and are denoted by ρ^o_{loc} and ρ^o_{oc}, respectively. An outer-connected open packing set of cardinality ρ^o_{oc} is called a ρ^o_{oc}-set of G.

Before providing the content of the thesis, we first note a few definitions and notations. All graphs considered here are finite, simple, undirected graphs. For basic definitions, refer to [5]. A subset S of E is called a *2-edge packing* if $N[e] \cap N[f] = \phi$ for all $e, f \in S$. The maximum cardinality of a 2-edge packing in G is called the *2-edge packing number* of G and is denoted by $P'_2(G)$. An edge $e = uv$ has vertices u and v that are saturated by e. Given a set of edges D, denote the set of vertices saturated by edges of D by V_D. The vertex $v \in V(G)$ is said to be a *maximum degree vertex* of a graph G if $deg(v) = \Delta(G)$. Denote a set of maximum degree vertices in a graph G by $V_{max}(G)$. The vertex v is said to be a *major independent vertex* if $deg(v) = \Delta(G)$ and $N(v)$ is an independent set. Denote the set of edges incident at the vertex v by $N_e(v)$. Note that $deg(v) = |N_e(v)|$. The i^{th} neighbor of a vertex in a graph G is defined by $N_i(v) = \{u : d(u, v) = i\}$. For a set $S \subseteq V(G)$, we define $N_S(v)$ by $N_S(v) = N(v) \cap S$. Denote $V^-_{[k]}(G) = \{v \in V(G) : \rho^o_e(G) - \rho^o_e(G - v) = k\}$, where $k \leq \Delta(G)$. An edge $e = uv$ is said to be a *support edge* if either u or v is a support vertex. An edge $e = uv$ is said to be a *double support edge* if both u and v are support vertices in G.

2 Literature Review

Meir and Moon [6] originated the concept of k-packing numbers. Further investigations on k-packing were done by Chang and Nemhauser [7, 8], Domke et al. [9], Topp and Volkmann [10]. Especially, various researchers for instances, the authors Burger et al. [11], Biggs et al. [12], Clark [13], Henning [14] and Rubalcaba et al. [15] have concerned their research on the 2-packing number. The concept of open packing of graphs as a special case of a 2-packing set was introduced Meir and Moon [6] and further investigations were done by Henning [16], Henning and Slater [17], Rall [18], Mohammadi et al. [19], Mojdeh and Samadi [20], Sahul Hamid and Saravanakumar [21–24].

There are many sectors where Packet Radio Networks (PRN) are used. The numerous writers who provided different PRN scheduling strategies, one may refer to Sen and Huson [25], Wang and Ansari [26]. In particular, some of the authors, for example Barrett et al. [27], Walczak and Wojciechowski [28] were schedule the PRN using graph coloring. In this way, the authors Cardoso et al. [4] introduced the *"Injective edge chromatic number"* (i-edge coloring) to assign the channels between the stations in order to avoid secondary interference in the Packet Radio Network(PRN) and obtained several results on it. Further studies on i-edge coloring were done by the authors Kostochka et al. [29] and Miao et al. [30]. It is obvious that an i-edge coloring of G partitioning the edge set $E(G)$ into edge subsets E_i having the property that no three edges in E_i to form a consecutiveness in G. Of course, studying the nature of these subsets is more useful. For instance, suppose we have the position to find the maximum number of transmission lines of the given PRN in which the secondary interference does not occur. This situation can be interpreted by defining the parameter, we call as *edge open*

packing sets in graphs.

The investigation of the impact of removing a vertex or an edge on any graph theoretic parameter has interesting applications in the setting of the network. The author Harary [31] introduced the concept of "Changing and Unchanging" and the authors Carrington et. al [32] give a survey of problems regarding to changing unchanging of the domination parameters. Moreover, the authors Chellali et al. [33], Khelifi and Chellali [34], Grobler and Mynhardt [35] worked an extensive examination of this kind of studies related to the ideas of domination and independence number. Following this, we have analyzed the changes taking place in the values of edge open packing number due to vertex or edge removal operations in a given graph.

Velammal [36] introduced the *domination subdivision number* by subdividing the edges in a graph G in order to increase the domination number of G and further studies on this number can be found in Aram et al. [37], Bhattacharya and Vijayakumar [38]. Haynes et al. [39] introduced the concept of total domination subdivision number $sd_{\gamma_t}(G)$ of a graph G. This kind of work was carried out for several parameters in graph theory, such as roman domination number, secure domination number, total domination number, independence number, 2-independence number and so on (one may refer to Atapour et al. [40], Divya Rashmi et al. [41], Haynes [42], Haynes et al. [43], Meddah et al. [44]). In this sequel, we introduce the *open packing subdivision number* as the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the open packing number of G and is denoted by $sd_{\rho^o}(G)$.

One can easily observe from the definition of an open packing set, each component of $\langle S \rangle$ is either K_1 or K_2, whereas the outside of S is not predictable. So one can study a variation of open packing by imposing suitable conditions on the outside of open packing sets. Actually, this type of studies related to domination and independence parameters have been carried out by Cyman [45], Mojdeh and Alishahi [46], Sahul Hamid et al. [47]. This information leads to introduce a new variation of open packing, namely outer-connected open packing sets in graphs. The generalized Jahangir graph $J_{n,m}$ was introduced Ali et al. [48]. Hereafter, several authors have been worked on the generalized Jahangir graph for determined the covering number and domination related parameters of $J_{n,m}$, one can see Angel and Amutha [49], Mtarneh et al. [50], Mojdeh and Ghameshlou [51]. From this motivation, we find the exact value the outer-connected open packing number of $J_{n,m}$ for all $n \geq 2$ and $m \geq 3$.

3 Objectives

1. Introduce and initiate the concept of edge open packing number of graphs.

2. Study the nature of packing parameters by vertex deletion, edge deletion, edge addition and subdivision of edges in graphs.

3. Investigate the variations of packing parameters such as open packing subdivision number and outer-connected open packing in graphs.

4 Organization of Thesis

The thesis consists of five chapters.

Chapter 1. Introduction and Basic Terminologies
Chapter 2. Edge Open Packing number of a Graph
Chapter 3. Criticality on Edge Open Packing
Chapter 4. Open Packing Subdivision number of Graphs
Chapter 5. Outer-connected Open Packing sets in Graphs

4.1 Introduction and Basic Terminologies

In **Chapter 1**, we present the basic terminologies and notation. Further, we just describe the results on packing and open packing that have already been discovered and will be used in the subsequent chapters.

4.2 Edge Open Packing number of a Graph

In **Chapter 2**, we introduce the edge open packing number and provide some bounds and characterization results.

Proposition 4.2.1. *Let P_n be a path of size $m \geq 2$. Then*

$$\rho_e^o(P_n) = \begin{cases} \frac{m+2}{2} & if\ m \equiv 2(mod\,4) \\ \lceil \frac{m}{2} \rceil & otherwise. \end{cases}$$

Proposition 4.2.2. *For the cycles C_n with size $m \geq 3$, we have*

$$\rho_e^o(C_n) = \begin{cases} \frac{m}{2} - 1 & if\ m \equiv 2(mod\,4) \\ \lfloor \frac{m}{2} \rfloor & otherwise. \end{cases}$$

Proposition 4.2.3. *Let $G = G_{n_1,n_2,...,n_k}$ be a complete multipartite graph with partition $(V_1, V_2, ..., V_k)$, where $|V_i| = n_i$, $1 \leq i \leq k$ and $n_1 \leq n_2 \leq ... \leq n_k$. Then $\rho_e^o(G) = n_k$.*

Proposition 4.2.4. *For wheels W_n with size m, we have $\rho_e^o(W_n) = \lfloor \frac{m}{4} \rfloor$.*

Proposition 4.2.5. *Let G be the Petersen graph. Then $\rho_e^o(G) = 3$.*

Now, we characterize graphs G for which $\rho_e^o(G) = 2$. Before going to characterize, we prove the following proposition.

Proposition 4.2.6. *For any graph G, we have $\rho_e^o(G) \geq \left\lceil \frac{diam(G)}{2} \right\rceil$.*

Theorem 4.2.7. *$\rho_e^o(G) = 2$ if and only if the following conditions are true*

(i) $2 \leq diam(G) \leq 4$

(ii) *G is $K_{1,s}$-free, where $s \geq 3$ and*

(iii) for any two non-adjacent edges $e_1 = uv$ and $e_2 = xy$ such that e_1 and e_2 have no common edge in G, every vertex in $V(G) \setminus \{u, v, x, y\}$ is adjacent to at least two vertices in the set $\{u, v, x, y\}$.

In the following theorems, we characterize the graphs G for which $\rho_e^o(G) = m - 1$ and $\rho_e^o(G) = m - 2$.

Theorem 4.2.8. *Let G be a graph with size $m \geq 3$. Then $\rho_e^o(G) = m - 1$ if and only if G is a graph obtained from a star $K_{1,m-1}$ by subdividing exactly one edge of $K_{1,m-1}$ by once.*

Next, we characterize graphs G for which $\rho_e^o(G) = m - 2$. For this, we use the following definitions and describe new family of graphs as follows.

Let $\mathscr{A}_1$, $\mathscr{A}_2$ and $\mathscr{A}_3$ be the families of graphs obtained from P_5, P_6 and P_7 respectively by attaching $r(\geq 0)$ number of pendant edges at each support vertex of P_5, P_6 and P_7.

Let $\mathscr{A}_4$ and $\mathscr{A}_5$ be the families of graphs obtained from C_3 and C_4 respectively by attaching $t(\geq 0)$ pendant edges at exactly one vertex of C_3 and exactly one vertex of C_4.

Let $\mathscr{A}_6$ be the family of graphs obtained from a star $K_{1,s}$, where $s \geq 3$ by subdividing exactly two edges of $K_{1,s}$ once at a time.

Let $\mathscr{A}_7$ be the family of graphs obtained from a star $K_{1,s}$, where $s \geq 3$ by attaching two pendant edges at exactly one of the pendant vertices of $K_{1,s}$.

Theorem 4.2.9. *Let G be a graph with size m. Then $\rho_e^o(G) = m - 2$ if and only if G belongs to one of the families in $\{\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3, \mathscr{A}_4, \mathscr{A}_5, \mathscr{A}_6, \mathscr{A}_7\}$.*

In the following proposition, we present an upper bound for ρ_e^o in terms of the size and the minimum degree.

Proposition 4.2.10. *Let G be a graph of size m. Then $\rho_e^o(G) \leq \frac{m}{\delta(G)}$.*

Next, we characterize r-regular graphs which attain the bound in the above proposition. For this purpose, we construct a family of r-regular graphs $\psi_{r,n}$ as follows.

Let $V(G) = X \cup Y \cup Z$ such that no two vertices of X or Y or Z are adjacent, where $|X| = \frac{n}{2} - \frac{n}{2r}$, $|Y| = \frac{n}{2}$ and $|Z| = \frac{n}{2r}$. Consider $(r, r-1)$-biregular graph with the vertex sets X and Y in which the degree of each vertex in X is r and the degree of each vertex in Y is $r - 1$. The existence of a biregular graph is possible by the condition that $|X|r = |Y|(r - 1)$. Now, join each vertex in Z to r vertices of Y such that $N(u) \cap N(v) = \phi$ for any two vertices $u, v \in Z$ and let G be the resultant graph. It is obvious that G is an r-regular graph with n vertices such that $n \equiv 0(mod\ 2r)$.

Theorem 4.2.11. *Let G be an r-regular graph. Then $\rho_e^o(G) = \frac{m}{r}$ if and only if $G \in \psi_{r,n}$.*

Theorem 4.2.12. *Let G be a graph with the size and the clique number ω. Then $\rho_e^o(G) \leq m - \frac{\omega(\omega-1)}{2} + 1$. Furthermore, $\rho_e^o(G) = m - \frac{\omega(\omega-1)}{2} + 1$ if and only if G is either K_n or a graph obtained from K_n by attaching any number of pendant edges at exactly one vertex of K_n.*

Proposition 4.2.13. *If G is a graph with $g(G) \geq 3$, then $\rho_e^o(G) \geq \left\lceil \frac{g(G)}{2} \right\rceil - 1$. Moreover, the equality holds if and only if G is either complete or a cycle of size m with $m \not\equiv 0(mod\ \ 4)$.*

Theorem 4.2.14. *For any graph G, we have $P_2'(G) \leq \rho_e^o(G) \leq \Delta(G)P_2'(G)$.*

Theorem 4.2.15. *Given any two positive integers $a, b \geq 1$ and an integer $k \geq 3$ with $\left\lceil \frac{b}{k} \right\rceil \leq a \leq b \leq ka$, there exists a graph G such that $P_2'(G) = a$, $\rho_e^o(G) = b$ and $\Delta(G) = k$.*

Next we discuss the edge open packing number for connected split graphs.

Lemma 4.2.16. *Let G be a connected split graph with split partition (V_1, V_2). Then V_1 contains at least one vertex v such that $deg(v) = \Delta(G)$.*

Theorem 4.2.17. *Let G be a connected split graph with the clique number $\omega(G)$. Then $\rho_e^o(G) \leq \Delta(G) - \omega(G) + 2$.*

Theorem 4.2.18. *Let G be a connected split graph. Then $\rho_e^o(G) = \Delta(G) - \omega(G) + 2$ if and only if G has a split partition (V_1, V_2) with the condition that for at least one vertex $u \in V_1$ with $deg(u) = \Delta(G)$, there exist at least one vertex $v \neq u$ in V_1 such that $N_{V_2}(u) \cap N_{V_2}(v) = \phi$.*

The following results shows the characterization of trees and unicyclic graphs with the edge open packing number is either 2 or 3.

Theorem 4.2.19. *Let T be a tree. Then $\rho_e^o(T) = 2$ if and only if T is either P_3 or P_4 or P_5.*

Proposition 4.2.20. *If $\rho_e^o(G) = 3$, then $diam(G) \leq 5$.*

Theorem 4.2.21. *Let T be a tree of order at least 4. Then $\rho_e^o(T) = 3$ if and only if either $T \cong P_6$ or $T \in \{B_{2,s} : 0 \leq s \leq 2\}$ or T is isomorphic to one of the graphs in $\{T_1, T_2, T_3, T_4, T_5, T_6\}$ shown in Figure 4.1.*

Proposition 4.2.22. *Let G be a unicyclic graph. If $H = \langle C_3 \rangle \subseteq G$ and $\rho_e^o(G) = 2$, then $\langle E(G) \setminus E(C) \rangle \cong P_2$ or P_3.*

Theorem 4.2.23. *Let G be a unicyclic graph. Then $\rho_e^o(G) = 2$ if and only if G is either one of the cycles C_4, C_5 and C_6 or isomorphic to one of the graphs H_i, $(1 \leq i \leq 4)$ shown in Figure 4.2.*

Note 4.2.24. *Let G be a unicyclic graph with C_t as an induced subgraph and $\rho_e^o(G) = 3$. Then $t \leq 7$. If $4 \leq t \leq 7$, then $2 \leq deg(u) \leq 3$ for any $u \in V(G)$. If $t = 3$, then $2 \leq deg(u) \leq 4$ for all $u \in V(C_3)$ and $2 \leq deg(u) \leq 3$ for all $u \in V(G) \setminus V(C_3)$.*

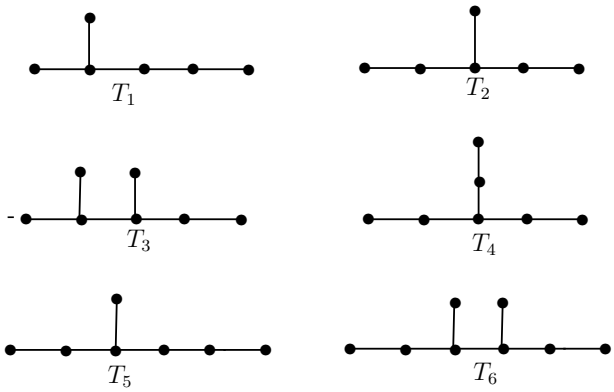

Figure 4.1: Trees T_1 to T_6.

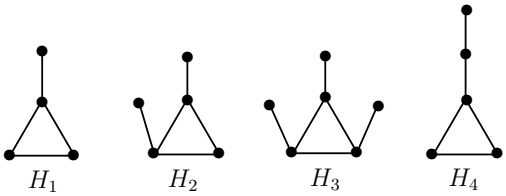

Figure 4.2: Graphs from H_1 to H_4.

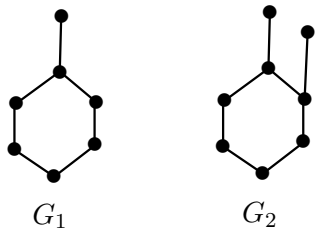

Figure 4.3: Graphs G_1 and G_2.

Theorem 4.2.25. *Let G be a unicyclic graph with the cycle C_t as an induced subgraph. Let $\rho_e^o(G) = 3$. Then $t = 7$ if and only if $G \cong C_7$.*

Theorem 4.2.26. *Let G be a unicyclic graph with the cycle C_6 as an induced subgraph. Then $\rho_e^o(G) = 3$ if and only if G is one of the graphs shown in Figure 4.3.*

Theorem 4.2.27. *Let G be a unicyclic graph with the cycle C_5 as an induced subgraph. Then $\rho_e^o(G) = 3$ if and only if G is one of the graphs shown in Figure 4.4.*

Theorem 4.2.28. *Let G be a unicyclic graph with the cycle C_4 as an induced subgraph. Then $\rho_e^o(G) = 3$ if and only if G is one of the graphs shown in Figure 4.5.*

Next, we characterize the unicyclic graphs with cycle $C_3 : (v_1v_2v_3v_1)$ for which $\rho_e^o(G) = 3$.

Theorem 4.2.29. *Let G ba a unicyclic graph with $C_3 : (v_1v_2v_3v_1)$ as an induced subgraph. Let $2 \leq diam(G) \leq 3$. Then $\rho_e^o(G) = 3$ if and only if $G \cong G_i$ for any i, $1 \leq i \leq 9$ shown in Figure 4.6.*

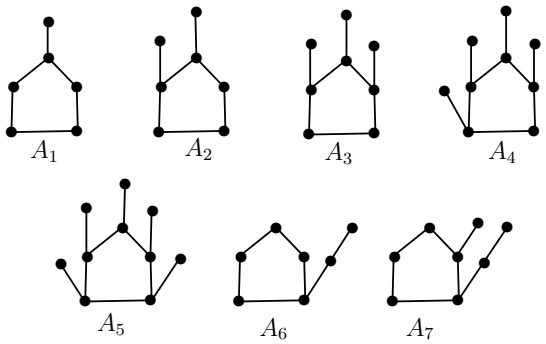

Figure 4.4: Graphs from A_1 to A_7.

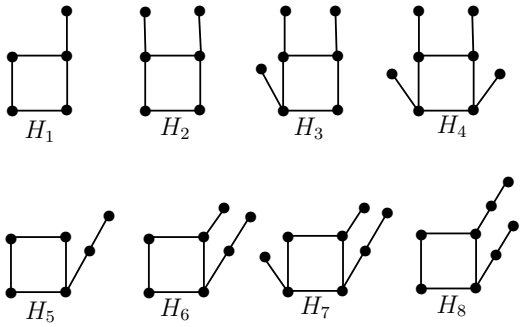

Figure 4.5: Graphs from H_1 to H_8.

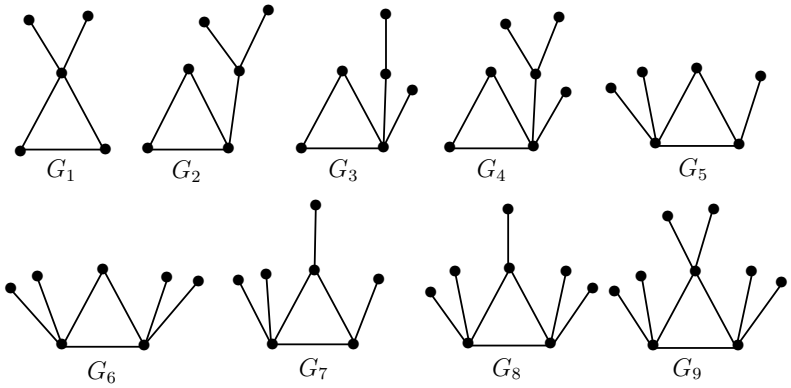

Figure 4.6: Graphs from G_1 to G_9.

Theorem 4.2.30. *Let G be a unicyclic graph with cycle $C_3 : (v_1v_2v_3v_1)$ as an induced subgraph. Let $diam(G) = 4$. Then $\rho_e^o(G) = 3$ if and only if G is isomorphic to one of the graphs in $\{J_1, J_2, ..., J_{10}\}$(Graphs from J_1 to J_{10} are shown in Figure 4.7).*

Theorem 4.2.31. *Let G be a unicyclic graph with cycle $C_3 : (v_1v_2v_3v_1)$ as an induced subgraph. Let $diam(G) = 5$. Then $\rho_e^o(G) = 3$ if and only if G is isomorphic to one of the graphs U_i, $(1 \leq i \leq 4)$, shown in Figure 4.8*

Proposition 4.2.32. *If G is a graph with at least one major independent vertex, then $\rho_e^o(G) \geq \Delta(G)$.*

Proposition 4.2.33. *For any unicyclic graph G, we have $\rho_e^o(G) \geq \Delta(G) - 1$.*

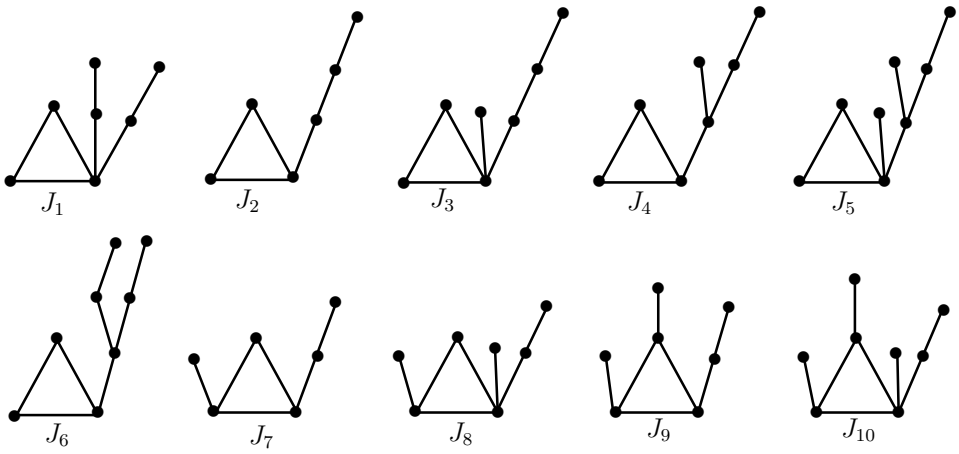

Figure 4.7: Graphs from J_1 to J_{10}.

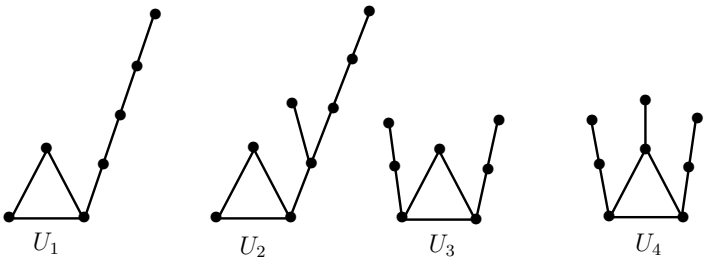

Figure 4.8: Graphs from U_1 to U_4.

Theorem 4.2.34. *Let T be a tree and let $k \geq 3 \in Z$ be an integer. Then $\rho_e^o(T) \geq \sum_{v \in S} deg(v)$ where $S \subseteq V(T)$ is a k-packing set of maximum cardinality.*

From Proposition 4.2.32, we observe that for any tree T, $\rho_e^o(T) \geq \Delta(T)$ as every maximum degree vertex in T is a major independent vertex. Next, we look for characterization of trees for which $\rho_e^o(T) = \Delta(T)$.

Theorem 4.2.35. *Let T be a tree with $\rho_e^o(T) = \Delta(T)$. Then $d(x, y) \leq 2$ for all $x, y \in V_{max}(T)$.*

Theorem 4.2.36. *Let T be a tree with $\rho_e^o(T) = \Delta(T)$. Then $|V_{max}(T)| \leq 3$.*

Theorem 4.2.37. *Let T be a tree with $|V_{max}(T)| = 3$. Then $\rho_e^o(T) = \Delta(T)$ if and only if $T \cong P_5$.*

Theorem 4.2.38. *Let T be a tree of order at least 2 with $|V_{max}(T)| = 2$. Then $\rho_e^o(T) = \Delta(T)$ if and only if T is isomorphic to either $B_{r,r}$ with $r \geq 0$ or T_1 or T_2, where T_1 is obtained from $B_{r,r}$ by adding exactly one pendant edge at any one of the pendant vertices of $B_{r,r}$ and T_2 is obtained from $B_{r,r}$ by adding exactly one pendant edge at exactly two pendant vertices in $B_{r,r}$ with distinct support.*

4.3 Criticality on Edge Open Packing

In **Chapter 3**, we investigate the fluctuation of the edge open packing number

through vertex removal and edge removal. Moreover, we examine the sets $V^-(G)$, $V^0(G)$, $E^-(G)$, $E^0(G)$ and $E^+(G)$ in this chapter.

Theorem 4.3.1. *Let G be a disconnected graph with components $G_1, G_2, ..., G_s$. Then $\rho_e^o(G) = \sum_{i=1}^{s} \rho_e^o(G_i)$.*

Observation 4.3.2. *For any graph G, $V(G) = V^0(G) \cup V^-(G)$, where*
$V^0(G) = \{v \in V(G) : \rho_e^o(G - v) = \rho_e^o(G)\}$
$V^-(G) = \{v \in V(G) : \rho_e^o(G - v) < \rho_e^o(G)\}$

Theorem 4.3.3. *For the path $P_n : (v_1, v_2, ..., v_n)$ on $n \geq 2$ vertices, we have*

(i) If $m \equiv 1(mod\ 4)$, then $V^0(P_n) = \{v_i : i \equiv 0$ or $3(mod\ 4)\}$

(ii) If $m \equiv 2(mod\ 4)$, then $V^0(P_n) = \{v_i : i \equiv 0(mod\ 4)\}$

(iii) If $m \equiv 3(mod\ 4)$, then $V^0(P_n) = \{v_i : i \equiv 0$ or $1(mod\ 4)\}$

(iv) If $m \equiv 0(mod\ 4)$, then $V^0(P_n) = V(P_n)$

and $V^-(P_n) = V(P_n) - V^0(P_n)$.

Theorem 4.3.4. *Let C_n be a cycle on $n \geq 3$ vertices. Then $V^0(C_n) = V(C_n)$ for all $n \geq 3$.*

Theorem 4.3.5. *(i) For a star graph $K_{1,m}$, $V(K_{1,m}) = V^-(K_{1,m})$*

(ii) For the Petersen graph G, $V^0(G) = V(G)$.

Theorem 4.3.6. *Let $G = K_{n_1,n_2,...,n_k}$ be a complete k-partite graph, which is not a star with partition set $(V_1, V_2, ..., V_k)$, $|V_i| = n_i$ for all $1 \leq i \leq k$ and $1 \leq n_1 \leq n_2 \leq ... \leq n_k$. Then $V(G) = V^0(G)$ if there exist at least two partitions V_i and V_j in G with $|V_i| = |V_j| = n_k$. For otherwise $V_k(G) = V^-(G)$ and $V_i(G) = V^0(G)$ for all $1 \leq i \leq k - 1$.*

It is obvious that any edge open packing set of $G-v$ is also an edge open packing set of G, we have $\rho_e^o(G) \geq \rho_e^o(G - v)$. Now, we give the necessary and sufficient condition for $v \in V^-(G)$.

Proposition 4.3.7. *Let G be a graph with $n \geq 2$ vertices. Then $v \in V^-(G)$ if and only if v is incident with at least one edge in every ρ_e^o-set of G.*

Corollary 4.3.8. *(i) Let G be a graph and let $v \in V(G)$. Then $\rho_e^o(G) = \rho_e^o(G - v)$ if and only if $v \notin V_D$ for some ρ_e^o-set D of G.*

(ii) Let D be a ρ_e^o-set of G. Then $V_D = V^-$ if and only if D is unique.

With the aid of Proposition 4.3.7, next we show that the difference between $\rho_e^o(G)$ and $\rho_e^o(G - v)$ is at most $\Delta(G)$.

Theorem 4.3.9. *Let G be a graph and let $v \in V(G)$. Then $\rho_e^o(G) - \rho_e^o(G-v) \leq \Delta(G)$.*

Lemma 4.3.10. *If $v \in V_{[t]}^-(G)$, then v is incident with at least t edges in every ρ_e^o-set of G.*

Theorem 4.3.11. *Let G be a graph and let $v \in V(G)$. Then $v \in V_{[\Delta(G)]}^-(G)$ if and only if the following are true.*

(i) *v is a major independent vertex.*

(ii) *for every ρ_e^o-set S in $G-v$ with $N(v) \cap V_S = \phi$ and $N_2(v) \cap V_S = \phi$.*

Proposition 4.3.12. *For any graph G, we have $|V^-(G)| \leq (\Delta(G)+1)P_2'(G)$ and this bound is sharp.*

Theorem 4.3.13. *Let G be a graph with $\Delta(G) \geq 2$. Then $|V^-(G)| = (\Delta(G)+1)P_2'(G)$ if and only if there exists a unique subset S of $V_{max}(G)$ fulfill the conditions*

(i) *$|S| = P_2'(G)$ and each vertex in S is a major independent vertex in G.*

(ii) *$d(u,v) \geq 4$ in G for all $u, v \in S$.*

Proposition 4.3.14. *Let G be a graph with $n \geq 3$ vertices. Then $V(G) = V^-(G)$ if and only if $G \cong K_{1,m}$.*

Proposition 4.3.15. *There is no graph G in which $\rho_e^o(G-v) = \rho_e^o(G) - \Delta(G)$ for all $v \in V(G)$.*

In the results that follow, we examine how the edge open packing number changes when a single edge is removed from the graph.

Observation 4.3.16. *For any graph G, $E(G) = E^-(G) \cup E^0(G) \cup E^+(G)$, where*
$E^-(G) = \{e \in E(G) : \rho_e^o(G-e) < \rho_e^o(G)\}$
$E^0(G) = \{e \in E(G) : \rho_e^o(G-e) = \rho_e^o(G)\}$
$E^+(G) = \{e \in E(G) : \rho_e^o(G-e) > \rho_e^o(G)\}$

Proposition 4.3.17. *Let P_n be a path on $m \geq 2$ edges. Then we have*

(i) *If $m \equiv 0(mod\,4)$, then $E^0(P_n) = \{e_i : i \equiv 0$ or $1(mod\,4)\}$ and $E^+(P_n) = \{e_i : i \equiv 2$ or $3(mod\,4)\}$.*

(ii) *If $m \equiv 1(mod\,4)$, then $E^-(P_n) = \{e_i : i \equiv 1(mod\,4)\}$, $E^0(P_n) = \{e_i : i \equiv 2$ or $0(mod\,4)\}$ and $E^+(P_n) = \{e_i : i \equiv 3(mod\,4)\}$.*

(iii) *If $m \equiv 2(mod\,4)$, then $E^-(P_n) = \{e_i : i \equiv 1$ or $2(mod\,4)\}$ and $E^0(P_n) = \{e_i : i \equiv 0$ or $3(mod\,4)\}$.*

(iv) *If $m \equiv 3(mod\,4)$, then $E^0(P_n) = E(P_n)$.*

Proposition 4.3.18. *For any cycle C_n, we have*

$$E(C_n) = \begin{cases} E^+(C_n) & if\ m \equiv 2\ or\ 3(mod\ 4) \\ E^0(C_n) & if\ m \equiv 0\ or\ 1(mod\ 4) \end{cases}$$

Proposition 4.3.19. *For a star $K_{1,m}$, we have $E(K_{1,m}) = E^-(K_{1,m})$.*

Proposition 4.3.20. *For the Petersen graph G, $E(G) = E^+(G)$.*

Proposition 4.3.21. *Let G be a complete k-partite graph with the partition set $(V_1, V_2..., V_k)$, which is not a star and $|V_i| = n_i$ for all $1 \leq i \leq k$ such that $1 \leq n_1 \leq n_2 \leq ... \leq n_k$. Then $E(G) = E^+(G)$ if $n_i = 1$ for all i; otherwise $E(G) = E^0(G)$*

If $\Delta(G) = 2$, then G is either path or cycle and from Propositions 4.3.17 and 4.3.18, we conclude that $\rho_e^o(G) - 1 \leq \rho_e^o(G - e) \leq \rho_e^o(G) + 1$.

Proposition 4.3.22. *Let G be a graph with $\Delta(G) \geq 3$ and let $e \in E(G)$. Then $\rho_e^o(G) - 1 \leq \rho_e^o(G - e) \leq \rho_e^o(G) + \Delta(G) - 2$.*

Theorem 4.3.23. *Let G be a connected graph of order $n \geq 3$. Then $e \in E^-(G)$ if and only if e belongs to every ρ_e^o-set D of G and $D \setminus \{e\}$ is a ρ_e^o-set of $G - e$.*

Corollary 4.3.24. *For any graph G, we have $|E^-(G)| \leq \rho_e^o(G)$ and the bound is sharp.*

Corollary 4.3.25. *Let G be a graph with $n \geq 3$ vertices. Then $E(G) = E^-(G)$ if and only if $G \cong K_{1,m}$.*

Theorem 4.3.26. *If G is a graph of order $n \geq 3$ and $e \in E(G)$. Then $e = uv \in E^+$ if and only if there exist at least one ρ_e^o-set D of $G - e$ satisfies $u, v \in V_D$ and one of the following conditions.*

(i) There exist at least one vertex x in $N(u) \cup N(v)$ such that $deg_{\langle D \rangle}\ x \geq 2$.

(ii) $deg_{\langle D \rangle}\ u \geq 2$ and $deg_{\langle D \rangle}\ v \geq 2$.

Corollary 4.3.27. *If e is a pendant edge, then $e \in E^0(G) \cup E^-(G)$. Moreover, $e \in E^-(G)$ if and only if $e \in D$ for every ρ_e^o-set D of G.*

Proposition 4.3.28. *Let G be a graph and $e = uv$ be a double support edge in G. Then $e \in E^0(G) \cup E^+(G)$.*

Theorem 4.3.29. *For any positive integers a, b, $c \geq 3$ and $b \geq c$ such that $a = b + c - 2$, there exists a graph G and an edge $e \in E(G)$ such that $\rho_e^o(G - e) = a$, $\rho_e^o(G) = b$ and $\Delta(G) = c$.*

4.4 Open Packing Subdivision number of Graphs

In **Chapter 4**, we initiate the parameter open packing subdivision number of graphs and we prove that $1 \leq sd_{\rho^o}(G) \leq 3$ for any graph G. Also, we characterize the graphs for which $sd_{\rho^o}(G) = 1$ and investigate sd_{ρ^o} for split graphs.

Convention: It is obvious that, if G is K_2, then the subdivision of the edge of G does not alter the value of $\rho^o(G)$ and so the parameter open packing subdivision is well-defined only for graphs having a component with at least three vertices. Henceforth, throughout this chapter, graphs are assumed to have a component with at least three vertices.

Proposition 4.4.1. *(i) For the path P_n on $n \geq 3$ vertices,*

$$sd_{\rho^o}(P_n) = \begin{cases} 3, & if\ n \equiv 2(mod\,4), \\ 2, & if\ n \equiv 3(mod\,4), \\ 1, & otherwise. \end{cases}$$

(ii) Let C_n be the cycle with $n \geq 3$ vertices, then

$$sd_{\rho^o}(C_n) = \begin{cases} 3, & if\ n \equiv 0(mod\,4), \\ 2, & if\ n \equiv 1(mod\,4), \\ 1, & otherwise. \end{cases}$$

Proposition 4.4.2. *Let G be a complete k-partite graph $(k \geq 2)$ with a partition set $(V_1, V_2, ..., V_k)$. If G is a star, then $sd_{\rho^o}(G) = 2$, otherwise, we have,*

$$sd_{\rho^o}(G) = \begin{cases} 3, & if\ k = 2, \\ 1, & if\ k \geq 3. \end{cases}$$

Proposition 4.4.3. *If G is the Petersen graph, then $sd_{\rho^o}(G) = 1$.*

Proposition 4.4.4. *Let G be a connected graph with $n \geq 3$ vertices and G' be the graph obtained by subdividing exactly one edge of G. Then we have, $\rho^o(G) \leq \rho^o(G') \leq \rho^o(G) + 1$.*

Theorem 4.4.5. *For any connected graph G with $n \geq 3$ vertices, we have $1 \leq sd_{\rho^o}(G) \leq 3$.*

Theorem 4.4.6. *Let G be a disconnected graph with the components $G_1, G_2, ..., G_r$. Then $sd_{\rho^o}(G) = min\{sd_{\rho^o}(G_i) : 1 \leq i \leq r\}$, where $|V(G_i)| \geq 3$ for at least one $i(1 \leq i \leq r)$.*

Corollary 4.4.7. *Let G be a disconnected graph. Then $sd_{\rho^o}(G) \geq 1$ if and only if the open packing subdivision number is at least one for at least one component in G.*

Corollary 4.4.8. *Let G be a disconnected graph with at least one component of order $n \geq 3$. Then $1 \leq sd_{\rho^o}(G) \leq 3$.*

Now, we categorize the graphs into three classes regarding the open packing subdivision number as follows.

Class A: Graphs with $sd_{\rho^o}(G) = 1$.
Class B: Graphs with $sd_{\rho^o}(G) = 2$.
Class C: Graphs with $sd_{\rho^o}(G) = 3$.

Theorem 4.4.9. *Let G be a connected graph with $n \geq 3$ vertices. Then G is in Class A if and only if there exists a ρ^o-set S in G such that S satisfies one of the following conditions:*

(i) $\langle S\rangle$ *has an isolated vertex.*

(ii) $\langle V-S\rangle$ *contains an edge* uv *with* $N(u)\cap S=\phi$ *and* $N(v)\cap S=\phi$.

Corollary 4.4.10. *If* $\rho^o(G)$ *is odd then* $sd_{\rho^o}(G)=1$.

Theorem 4.4.11. *If* G *has a strong support, then* $sd_{\rho^o}(G)\leq 2$.

The converse of Theorem 4.4.11 need not be true because of the graphs $G=C_3\circ K_1$ and P_7 has no strong support but $sd_{\rho^o}(G)=1$ and $sd_{\rho^o}(P_7)=2$.

For a ρ^o-set S of a connected split graph G, we just observe the following.

Observation 4.4.12. (i) *If* $S\cap V_1\neq\phi$, *then* $|S|\leq 2$.

(ii) *If* $|S|\geq 3$, *then* $S\subseteq V_2$ *and* $sd_{\rho^o}(G)=1$.

(iii) *If* $\langle S\rangle=K_2$ *for any* ρ^o*-set* S *in* G, *then* $sd_{\rho^o}(G)=2$.

Theorem 4.4.13. *Let* G *be a connected split graph with a split partition* (V_1,V_2). *Then* $sd_{\rho^o}(G)\leq 2$. *Furthermore,* $sd_{\rho^o}(G)=2$ *if and only if* $N(u)\cap N(v)\neq\phi$ *for each* $u,v\in V_2$ *and* $\delta(G)=1$.

The following theorem is a direct consequence of Theorem 4.4.6 and Corollary 4.4.8.

Theorem 4.4.14. *Let* F *be a forest with at least one component of order* $k\geq 3$. *Then*

(i) $sd_{\rho^o}(F)=1$ *if and only if at least one component of* F *of order more than 2 belongs to Class A.*

(ii) $sd_{\rho^o}(F)=3$ *if and only if each component of order at least 3 in* F *should belong to Class C.*

4.5 Outer-connected Open Packing sets in Graphs

In **Chapter 5**, we introduce the outer-connected open packing number of graphs and we provide bounds and characterization results. Moreover, we compare the outer-connected open packing number to other parameters in graph theory such as the open packing number, the lower open packing number and the 2-packing number, along with some realization theorems.

Proposition 4.5.1. *(i) Let* P_n *be a path on* $n\geq 2$ *vertices. Then*

$$\rho^o_{loc}(P_n)=\rho^o_{oc}(P_n)=\begin{cases}2 & if\ n\leq 4\\ 3 & if\ n=5\\ 4 & otherwise\end{cases}$$

(ii) If C_n is a cycle on $n \geq 4$ vertices, then $\rho^o_{loc}(C_n) = \rho^o_{oc}(C_n) = 2$.

(iii) The values of both ρ^o_{loc} and ρ^o_{oc} are equal to 2 for the complete bipartite graphs and the Petersen graph.

(iv) For the complete k-partite graphs, where $k \geq 3$, the values of ρ^o_{loc} and ρ^o_{oc} are equal to 1.

Theorem 4.5.2. *If G is a connected graph of order n, then $1 \leq \rho^o_{loc}(G) \leq \rho^o_{oc}(G) \leq n$. Furthermore,*

(i) $\rho^o_{oc}(G) = 1$ if and only if diam$(G) \leq 2$ and for every edge $uv \in E(G)$ either uv lies in a triangle or $\omega(G - \{u, v\}) \geq 2$, where $\omega(H)$ denotes the number of components in H.

(ii) $\rho^o_{oc}(G) = n$ if and only if G is either K_1 or K_2.

Theorem 4.5.3. *Let G be a connected graph of order n. Then $\rho^o_{oc}(G) = n - 1$ if and only if G is P_3.*

Theorem 4.5.4. *Let G be a connected graph. If $\rho^o_{oc}(G) = 1$, then $\rho^o(G) \leq 2$.*

Next, we present an upper bound for the outer-connected open packing number in terms of the connected domination number of a graph.

Proposition 4.5.5. *If G is a connected graph of order $n \geq 3$ with $\delta(G) \geq 2$, then $\rho^o_{oc}(G) \leq n - \gamma_c(G)$ and the bound is sharp.*

In the following, we establish an upper bound for the outer-connected open packing number for the family of trees in terms of the number of support vertices and characterize the trees achieving this bound. Before we state the theorem that, we first need the following proposition.

Proposition 4.5.6. *Let G be a connected graph of order at least 3 and S be an ocop-set of G. If S contains a cut vertex v of G, then v is a support vertex of G and $\omega(G - v) = 2$, where $\omega(H)$ is the number of components in H.*

Theorem 4.5.7. *Let T be a tree of order at least 3. Then $\rho^o_{oc}(T) \leq 2s$, where s is the number of support vertices of T. Furthermore, the equality holds if and only if every support vertex of T has degree two and the distance between any two support vertices in T is at least 3.*

Theorem 4.5.8. *Let G be a caterpillar of order at least 3. Then $\rho^o_{oc}(G) = 2s$ if and only if G is a path of order n, where $n \notin \{4, 5\}$.*

Next, we finding a possible relationship between *outer-connected open packing number* and the open packing number, the lower open packing number and the 2-packing number, along with some realization theorems. Since for an arbitrary connected graph, every outer-connected open packing set is also an open packing set, we have the inequality $\rho^o_{oc} \leq \rho^o$. However, the lower open packing number ρ^o_L is incomparable with both ρ^o_{oc} and ρ^o_{loc} which we later prove. Now, we begin with the following theorem that proves that the difference between the parameters ρ^o and ρ^o_{oc} is arbitrarily large.

Theorem 4.5.9. *For any non-negative integer $a \geq 0$, there exists a connected graph G such that $\rho^o(G) - \rho^o_{oc}(G) = a$.*

Theorem 4.5.10. *For any two positive integers $a \geq 2$ and b such that $a \leq b$, there exists a connected graph G such that $\rho^o_{oc}(G) = a$ and $\rho^o(G) = b$.*

Theorem 4.5.11. *Given any two positive integers a and b such that $a \leq b$, there exists a connected graph G such that $\rho^o_{loc}(G) = a$ and $\rho^o_{oc}(G) = b$.*

Theorem 4.5.12. *ρ^o_L and ρ^o_{oc} are incomparable.*

Corollary 4.5.13. *ρ^o_L and ρ^o_{loc} are incomparable.*

Theorem 4.5.14. *$\rho(G)$ and $\rho^o_{oc}(G)$ are incomparable.*

Theorem 4.5.15. *For a generalized Jahangir graph $J_{n,m}$ with $n \geq 2$ and $m \geq 3$, we have*

$$\rho^o_{oc}(J_{n,m}) = \begin{cases} \left\lceil \frac{m-1}{2} \right\rceil + 1 & if\ n = 2 \\ m+1 & if\ n = 3 \\ 2m & if\ n \geq 4 \end{cases}$$

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1. S. Saravanakumar and **C. Gayathri**, Outer-connected open packing sets in graphs, *Asian-Eur. J. Math.*, **15** (5) (2022) 2250083.

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5. **C. Gayathri** and S. Saravanakumar, Trees and Unicyclic graphs with Packing number Two, *In Proc. of the International conference on Advanced Mathematical Modeling and Computational Techniques (ICAMMCT-2021)*, (2021) 437-444.